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Critical behaviour of systems with cubic anisotropy and random impurities

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Abstract. Renormalisation group recursion relations are obtained to two-loop order for an n-component Landau-Ginzburg-Wilson model containing both hypercubic anisotropy and quenched random impurities. The fixed points are enumerated and their stability properties analysed, using the ε expansion about four spatial dimensions to second order. The existence of a new fixed point corresponding to an anisotropic impure system is reported. It is found that both isotropic and anisotropic pure systems are unstable to the addition of random impurities for $n < n_R \equiv 4-4\varepsilon+O(\varepsilon^2)$, while both pure and random isotropic systems are unstable to anisotropy for $n > n_C \equiv 4-2\varepsilon+O(\varepsilon^2)$. Singularities in the ε expansions for the exponents of the random fixed points are shown to be associated with the onset of focal behaviour (the acquisition of complex eigenvalues). These, together with the slow convergence of the expansions lead us to doubt the reliability of extrapolations to three dimensions.

1. Introduction

It has long been known that the critical behaviour of an n-component spin system having either cubic anisotropy or quenched random impurities may differ from that of the corresponding pure, isotropic system. The analysis of these effects continues to attract both theoretical and experimental interest.

Using the description of critical behaviour afforded by phenomenological Hamiltonians of the Landau-Ginzburg-Wilson type, Aharony (1973, 1976) investigated the effect of cubic anisotropy in pure systems by renormalisation group and ε-expansion methods. (As usual, we define $\varepsilon = 4-d$, where d is the spatial dimensionality of the system.) Calculations to second order of the expansion show that the isotropic behaviour, controlled by the Heisenberg fixed point of the renormalisation group, is unstable to cubic perturbations when n exceeds a critical value $n_C = 4-2\varepsilon+O(\varepsilon^2)$. When the anisotropy is weak, critical behaviour is controlled by a new stable fixed point. Under certain conditions, however, it appears that strong anisotropy can give rise to a first-order transition (Wallace 1973, Rudnick 1978). It is obviously important to know the value of n_C in three dimensions. Using a Padé approximant to third-order results of Ketley and Wallace (1973), Aharony (1976) estimates $n_C \approx 3.128$ at $\varepsilon = 1$. On the other hand, Yalabik and Houghton (1977) obtain $n_C \approx 2.3$, using the approximate recursion relations due to Kadanoff *et al* (1976). The question of whether $n_C(d=3)$

lies above or below 3 is still unresolved. Recently, it has become apparent that cubic anisotropy may be of special importance in interpreting experimental results on systems undergoing structural phase transitions (Müller 1984, Müller *et al* 1984, Fossheim 1984).

The effect of quenched, random impurities on the critical behaviour of isotropic spin systems has been studied by a number of authors. For $n>1$, Lubensky (1975) and Grinstein and Luther (1976) have identified a new renormalisation group fixed point appropriate to impure systems, having its own characteristic set of critical exponents. The pure Heisenberg fixed point is unstable to the introduction of random impurities when n lies below a critical value $n_R = 4-4\varepsilon + O(\varepsilon^2)$. This coincides with the value of n at which the specific heat exponent α vanishes in the pure, isotropic system, in agreement with the well known argument of Harris (1974) that critical behaviour should be unaffected by random impurities when α is negative. For systems with Ising symmetry ($n=1$), the pure behaviour is also unstable, but the renormalisation group equations exhibit a special degeneracy. As first shown by Khmel'nitskii (1975, see also Shalaev 1977) critical exponents, characteristic of a further fixed point, can be obtained as power series in $\varepsilon^{1/2}$. It has also proved possible to study models of systems in which random impurities have long-range correlations (Weinrib and Halperin 1983) or are extended objects (Dorogovtsev 1980, Boyanovsky and Cardy 1982, Lawrie and Prudnikov 1984). In this paper, however, we consider only uncorrelated point impurities.

In real magnetic systems with cubic anisotropy, one may in general expect random impurities always to be present in some degree, and the asymptotic critical behaviour will be determined by the outcome of the competition between the two effects. Equally, of course, one must in principle include the possibility of numerous other effects, such as dipolar interactions, lattice compressibility, etc, but we shall not aim here for full generality. Nor, indeed, are we able to address the question of whether departures from pure isotropic behaviour due to any of these effects will, in specific materials, be experimentally resolvable. Our purpose is to examine the renormalisation group structure of the n-component model containing both cubic anisotropy and random impurities, to enumerate the fixed points and to establish their stability properties as functions of n and d. The case $n=2$ has in fact been investigated recently by Yamazaki *et al* (1985), who discovered a degeneracy analogous to that which, as mentioned above, occurs in isotropic systems for $n=1$. They showed that in this case too an $\varepsilon^{1/2}$ expansion is appropriate, and we confirm their results up to a minor numerical difference. More generally, using recursion relations which for all n are valid to order ε^2, we identify a new fixed point corresponding to a system with both random point impurities and cubic anisotropy, and we record the values of its exponents to order ε^2. It seems, however, that this fixed point is never completely stable. If n is sufficiently close to 4, and if ε is sufficiently small, we find the following: the pure, isotropic fixed point is stable (and all others unstable in at least one direction) for $n_R < n < n_C$; for $n > n_C$, the pure cubic fixed point is stable, while for $n < n_R$ the random isotropic fixed point is stable. Unfortunately, the appearance of singularities (which we identify and discuss) in the ε expansion of critical exponents and the poor convergence of these expansions away from the singularities renders extremely hazardous any attempt to extend these assertions beyond the immediate neighbourhood of the point $n=d=4$. In particular, little can be said with confidence about the point $n=d=3$.

The model we study and its renormalisation group recursion relations are presented in § 2. In § 3, we examine the various fixed points of these relations and in § 4 we summarise and discuss our main findings.

2. Landau–Ginzburg–Wilson Hamiltonian and recursion relations

The Landau-Ginzburg-Wilson Hamiltonian for a system with cubic anisotropy and random point impurities may be written as

$$\mathscr{H} = -\tfrac{1}{2}\int \mathrm{d}^d x\left(|\nabla\phi(\boldsymbol{x})|^2 + \tilde{r}(\boldsymbol{x})\phi^2(\boldsymbol{x}) + u\phi^4(\boldsymbol{x}) + v\sum_{\alpha=1}^{n}\phi_\alpha^4(\boldsymbol{x})\right) \tag{2.1}$$

where $\phi(\boldsymbol{x}) = (\phi_1(\boldsymbol{x}), \ldots, \phi_n(\boldsymbol{x}))$ is the n-component order-parameter field, and ϕ^2 and ϕ^4 denote $\Sigma_{\alpha=1}^{n}\phi_\alpha\phi_\alpha$ and $(\phi^2)^2$ respectively. The isotropic and anisotropic coupling constants are respectively u and v, and the function $\tilde{r}(\boldsymbol{x})$ is given by

$$\tilde{r}(\boldsymbol{x}) = r + \rho(\boldsymbol{x}) \tag{2.2}$$

where r is assumed to be proportional to $T - T_c$ (with T_c the critical temperature) and $\rho(x)$ represents the impurity distribution. Evidently, $\rho(x)$ can be regarded as describing local, impurity-induced variations in T_c. We assume it to have a Gaussian distribution of variance Δ:

$$\langle\rho(\boldsymbol{x})\rho(\boldsymbol{x}')\rangle = \Delta\,\delta^d(\boldsymbol{x}-\boldsymbol{x}'). \tag{2.3}$$

Of course, non-Gaussian corrections to this distribution will normally be present, as will impurity-induced contributions to u and v, but these are all irrelevant, in the renormalisation group sense, and do not affect asymptotic critical behaviour (see, e.g., Lubensky 1975). Clearly, for $n = 1$, the isotropic and anisotropic interactions in (2.1) are identical, and the parameter v is redundant.

To obtain correlation functions and hence renormalisation group recursion relations, it is necessary to average over the quenched impurity distribution (2.3). This may be done either by using the replica trick (see, e.g., Grinstein and Luther 1976) or by direct averaging (Lubensky 1975). We have actually used the latter method, but the results should be identical. In treating recursion relations to second order in ε, we have used the large b limit (where b is the change in length scale in a single step of renormalisation) as described by Bruce *et al* (1974). An advantage of this method is that one can neglect all mass insertions in the perturbation series, which do not contribute to the final expressions for fixed points and critical exponents.

The recursion relations are

$$\begin{aligned} r' = b^{2-\eta}\{&r + [4(n+2)u - \Delta + 12v]A(r) \\ &-[32(n+2)u^2 - 8(n+2)u\Delta + \Delta^2 + 96v^2 + 192uv - 24\Delta v]B(r)\} \end{aligned} \tag{2.4a}$$

$$\begin{aligned} u' = b^{\varepsilon-2\eta}\{&u - [4(n+8)u^2 - 6u\Delta + 24uv]K_4(\ln b)(1+\varepsilon/2\ln b) \\ &+[16(n^2+6n+20)u^3 - 12(n+8)u^2\Delta + 9u\Delta^2 \\ &+144(n+4)u^2v + 432uv^2 - 72u\Delta v](K_4\ln b)^2 \\ &+[32(5n+22)u^3 - 48(n+5)u^2\Delta + 21u\Delta^2 \\ &+1152u^2v + 288uv^2 - 216u\Delta v]K_4^2(\ln b)(1+\ln b)\} \end{aligned} \tag{2.4b}$$

$$\begin{aligned} \Delta' = b^{\varepsilon-2\eta}\{&\Delta - [8(n+2)u\Delta - 4\Delta^2 + 24\Delta v]K_d(\ln b)(1+\varepsilon/2\ln b) \\ &+[5\Delta^3 - 24(n+2)u\Delta^2 + 48(n+2)^2u^2\Delta - 72\Delta^2 v + 432\Delta v^2 \\ &+288(n+2)u\,\Delta v](K_4\ln b)^2 + [11\Delta^3 - 48(n+2)u\Delta^2 \\ &+96(n+2)u^2\Delta - 144\Delta^2 v + 288\Delta v^2 + 576u\Delta v]K_4^2(\ln b)(1+\ln b)\} \end{aligned} \tag{2.4c}$$

$$v' = b^{\varepsilon-2\eta}\{v - [36v^2 + 48uv - 6\Delta v]K_d(\ln b)(1+\varepsilon/2\ln b)$$
$$+9[64u^2v + 96uv^2 + 48v^3 - 12\Delta v^2 + \Delta^2 v - 16u\Delta v](K_4 \ln b)^2$$
$$+3[32(n+14)u^2v + 768uv^2 + 288v^3 - 96\Delta v^2$$
$$+7\Delta^2 v - 8(n+14)u\Delta v]K_4^2(\ln b)(1+\ln b)\} \qquad (2.4d)$$

where $K_d = 2\pi^{d/2}/(2\pi)^d\Gamma(d/2)$ and the Fisher exponent is given by

$$\eta = K_d^2[8(n+2)u^2 - 2(n+2)u\Delta + 48uv + 1/4\Delta^2 + 24v^2 - 6\Delta v]. \qquad (2.5)$$

The quantities $A(r)$ and $B(r)$ are the usual one- and two-loop integrals

$$A(r) = \int \frac{d^d k}{(2\pi)^d}(k^2+r)^{-1} \qquad (2.6)$$

$$B(r) = \int \frac{d^d k_1\, d^d k_2}{(2\pi)^{2d}}[(k_1^2+r)(k_2^2+r)((k_1+k_2)^2+r)]^{-1} \qquad (2.7)$$

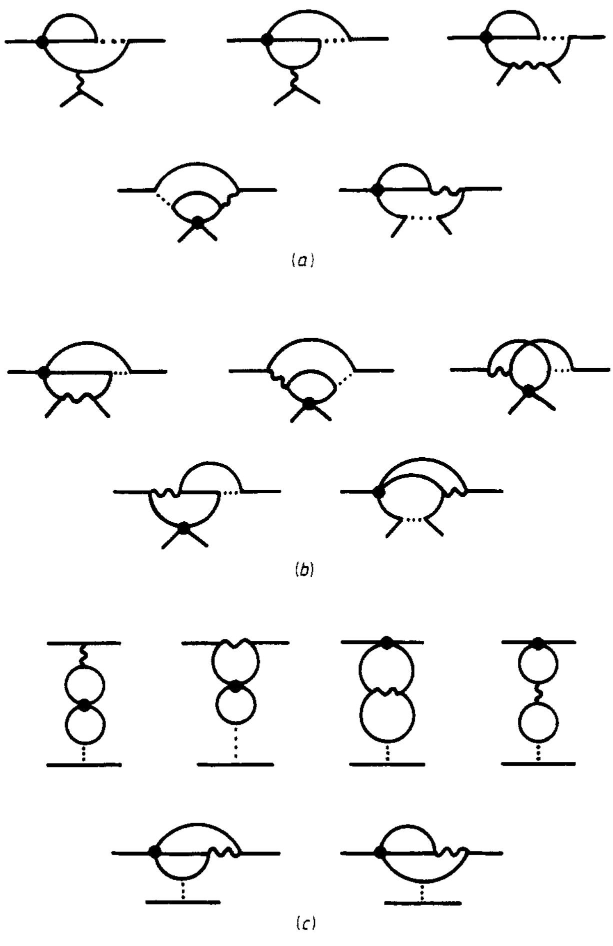

Figure 1. Diagrams proportional to $u\Delta v$ which contribute to the renormalisation of (a) u, (b) v and (c) Δ. The isotropic interaction is represented by a wavy line, the anisotropic interaction by a dot and the effective vertex arising from the impurity average by a dotted line.

with the ranges of integration given by $b^{-1} \leq |k| \leq 1$. These recursion relations are, of course, extensions of those given by Aharony (1973) for pure cubic systems and by Lubensky (1975) for impure, isotropic systems. The terms which have not been calculated previously are those proportional to $u\Delta v$, represented diagrammatically in figure 1. There are also terms proportional to Δv, Δv^2 and $\Delta^2 v$ which have not previously arisen: these however are simply obtained by setting $n=1$ in the coefficients of corresponding terms proportional to $u\Delta$, $u^2\Delta$ and $u\Delta^2$.

3. Fixed points and their stability properties

3.1. Fixed points

As always, the recursion relations (2.4) possess a trivial Gaussian fixed point, $u^* = v^* = \Delta^* = 0$. There are also fixed points with $\Delta^* < 0$, corresponding to the unphysical situation of an impurity distribution with negative variance. Since no renormalisation group flows cross the surface $\Delta = 0$, these unphysical points may safely be ignored. There are eight remaining non-trivial fixed points, which we denote as follows:

PH: pure isotropic Heisenberg fixed point ($u^* \neq 0$; $v^* = \Delta^* = 0$)
PC: pure cubic fixed point ($u^* \neq 0$; $v^* \neq 0$; $\Delta^* = 0$)
RH: random Heisenberg fixed point ($u^* \neq 0$; $v = 0$; $\Delta^* \neq 0$)
RC: random cubic fixed point ($u^* \neq 0$; $v^* \neq 0$; $\Delta^* \neq 0$)
PI: pure Ising fixed point ($u^* = 0$; $v^* \neq 0$; $\Delta^* = 0$)
RI: random Ising fixed point ($u^* = 0$; $v^* \neq 0$; $\Delta^* \neq 0$)
RH($n=1$): degenerate random Ising fixed point ($(u+v)^* \neq 0$; $\Delta^* \neq 0$)
RC($n=2$): degenerate random cubic fixed point ($u^* \neq 0$; $v^* \neq 0$; $\Delta^* \neq 0$).

Of these, the first four may be said to represent generic types of behaviour, while the last four arise from special circumstances. The PI and RI fixed points lie in the surface $u=0$ where, for any n, the model (2.1) consists of n decoupled Ising-like systems. At $n=1$, the expansion of u^* and Δ^* for the RH fixed point in integer powers of ε is singular, but one can instead obtain an expansion in powers of $\varepsilon^{1/2}$, denoted by RH($n=1$). Naturally, this fixed point has the properties associated with a random Ising system. Moreover since, as noted above, u and v are not independent parameters for $n=1$, it essentially coincides with the RI fixed point. At $n=2$, the RC fixed point is similarly ill defined as a power series in ε, but again the fixed point RC($n=2$) may be obtained as a series in $\varepsilon^{1/2}$. Conceivably, a suitable resummation might reveal that RC($n=2$) is in fact the limit as $n \to 2$ of RC, and similarly for RH($n=1$), but we are unable to show this in detail.

The random cubic fixed point is a new result of the present study. Its coordinates (within the renormalisation scheme used here) are

$$u^{\mathrm{RC}} = \frac{\varepsilon}{24(n-2)K_d}\left(1 - \frac{89n^2+728n-1600}{864(n-2)^2}\varepsilon\right) \tag{3.1}$$

$$\Delta^{\mathrm{RC}} = \frac{(4-n)\varepsilon}{12(n-2)K_d}\left(1 - \frac{703n^3-3396n^2+7392n-6400}{864(n-2)^2(4-n)}\varepsilon\right) \tag{3.2}$$

$$v^{\mathrm{RC}} = \frac{(n-4)\varepsilon}{72(n-2)K_d}\left(1 + \frac{487n^3-3252n^2+9120n-8704}{864(n-2)^2(n-4)}\varepsilon\right) \tag{3.3}$$

and its static critical exponents may be found using standard scaling relations from the Fisher exponent η and the correlation length exponent ν which are

$$\eta = \frac{\varepsilon^2}{5184(n-2)^2}(75n^2 - 168n - 96) + O(\varepsilon^3) \tag{3.4}$$

$$\nu = \tfrac{1}{2} + \frac{(5n-8)}{48(n-2)}\varepsilon + \frac{1289n^3 - 9612n^2 + 20\,064n - 12\,416}{41\,472(n-2)^3}\varepsilon^2 + O(\varepsilon^3). \tag{3.5}$$

3.2. *Stability of generic fixed points near $n = d = 4$*

Critical exponents of the PH, PC and RH fixed points are recorded by many authors (see, e.g., Wilson and Kogut 1974, Ma 1976, Brezin *et al* 1976, Aharony 1976, Yamazaki 1978) using the first few orders of the ε expansion. In order to determine which set of exponents controls asymptotic critical behaviour for given values of n and d, we must ascertain which of the fixed points is stable under the renormalisation group and will therefore attract flows from a large region of the parameter space. We consider first the four generic fixed points PH, PC, RH and RC. In the neighbourhood of a fixed point u_i^*, (with $(u_1, u_2, u_3) \equiv (u, \Delta, v)$), the recursion relations may be linearised in the form

$$(u_i' - u_i^*) = \sum_j A_{ij}(b)(u_j - u_j^*) \tag{3.6}$$

and the matrix A has eigenvalues of the form b^{y_i}, where y_i are universal exponents, which we obtain to order ε^2. Evidently, since $b > 1$, a negative exponent indicates an eigenperturbation against which the fixed point is stable, while a positive exponent indicates instability. Usually the exponents are obtained in the form

$$y(n, \varepsilon) = a(n)\varepsilon + b(n)\varepsilon^2 + O(\varepsilon^3) \tag{3.7}$$

where the functions $a(n)$ and $b(n)$ are exactly known. We need to discover where, if at all, such an exponent changes sign. Clearly, an estimate of a zero, $\bar{n}(\varepsilon)$, of y such that $b(\bar{n}) = -a(\bar{n})/\varepsilon$ and $a(\bar{n}) \neq 0$ will be highly unreliable, unless it can be supported by information about late terms of the expansion, and no such information is available to us. For this reason, we are suspicious of numerical estimates given by Yamazaki (1978) concerning relative stabilities of the pure and random isotropic fixed points, which we have cause to reconsider below. When ε is sufficiently small, the sign of $y(n, \varepsilon)$ is the same as that of $a(n)$ and the expansion (3.7) provides reliable evidence of a change of sign only if there is an n_0 such that $a(n_0) = 0$ and $a'(n_0) \neq 0$. Unless $b(n)$ is singular at n_0, we may then obtain an estimate of the zero, $\bar{n}(\varepsilon)$ in the form

$$\bar{n}(\varepsilon) = n_0 - [b(n_0)/a'(n_0)]\varepsilon + O(\varepsilon^2). \tag{3.8}$$

Only by means of such a systematic expansion can one guarantee that the stability properties obtained will be consistent with general features of the recursion relations. Namely, because of the polynomial character of the functions controlling the flow, changes in stability as n and ε vary can come about only when two fixed points pass through each other. Approximations, other than a systematic expansion in ε will in general yield inconsistent estimates of $\bar{n}$ when applied to the coordinates and eigenvalues of the two fixed points which are supposed to coincide.

Flows connecting the four fixed points are shown schematically in figure 2. The direction of flow along each side of the square is determined by the appropriate exponents of the two fixed points at its ends, one of which must be positive and the other negative unless the fixed points coincide. Horizontal flows in figure 2 reflect the importance of cubic anisotropy and the exponents will be denoted by y_v^i for the ith fixed point. Likewise, vertical flows reflect the importance of random impurities, and the exponents will be labelled $y_\Delta{}^i$. However, this notation does not necessarily imply that, for example, the eigenvector corresponding to y_v^i is exactly parallel to the v axis. Each fixed point also possesses an eigenvalue y_u^i which is always negative (insofar as no zero of the form (3.8) exists), so that the whole of figure 2 is stable against perturbations in u. The stabilities are most reliably obtained in the neighbourhood of $n = d = 4$. Consider first the flows in the v direction. To order ε^2, the exponents are

$$y_v^{\mathrm{PH}} = \frac{(n-4)}{(n+8)}\varepsilon + \frac{5n^2+14n+152}{(n+8)^3}\varepsilon^2 \tag{3.9}$$

$$y_v^{\mathrm{RH}} = \frac{(n-4)}{4(n-1)}\varepsilon - \frac{n(71n^2-388n-160)}{512(n-1)^3}\varepsilon^2 \tag{3.10}$$

$$y_v^{\mathrm{PC}} = \frac{(4-n)}{3n}\varepsilon + \frac{(n-1)(19n^3-72n^2-660n+848)}{81n^3(n+2)}\varepsilon^2 \tag{3.11}$$

$$y_v^{\mathrm{RC}} = \frac{(4-n)}{6(n-2)}\varepsilon + \frac{(4333n^4-44\,068n^3+144\,960n^2-196\,864n+96\,256)}{5184(5n-8)(n-2)^3}\varepsilon^2 \tag{3.12}$$

of which (3.10) and (3.12) are given here for the first time. All of these vanish at $n = n_{\mathrm{C}} = 4 - 2\varepsilon + \mathrm{O}(\varepsilon^2)$, and at this value the fixed points PH and PC coincide as do RH and RC. For $n < n_{\mathrm{C}}$, therefore, the isotropic fixed points PH and RH are stable against cubic perturbations, whereas they are unstable for $n > n_{\mathrm{C}}$. This leads to the satisfactory (though apparently not inevitable) conclusion that cubic anisotropy is relevant for both pure and impure systems in the same range of values of n and d.

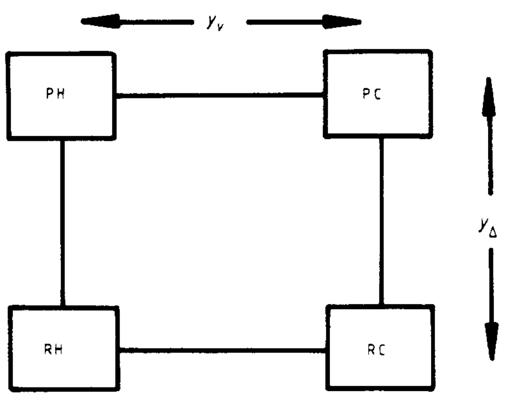

Figure 2. Schematic flow diagram exhibiting the pure Heisenberg (PH), pure cubic (PC), random Heisenberg (RH) and random cubic (RC) fixed points and the special trajectories connecting them. Horizontal flows are to the right (left) if anisotropy is relevant (irrelevant), according to the signs of the eigenvalues y_v^i. Vertical flows are downwards (upwards) if random impurities are relevant (irrelevant) as determined by the signs of y_Δ^i. The disposition of fixed points in the figure does not necessarily reflect the signs of v^* and Δ^*.

Turning to flows in the Δ direction, we have

$$y_\Delta^{PH} = \frac{(4-n)}{(n+8)}\varepsilon - \frac{(n+2)(13n+44)}{(n+8)^3}\varepsilon^2 \tag{3.13}$$

$$y_\Delta^{RH} = \frac{(n-4)}{4(n-1)}\varepsilon + \frac{n(-35n^3+1932n^2-3840n+512)}{512(5n-8)(n-1)^3}\varepsilon^2 \tag{3.14}$$

$$y_\Delta^{PC} = \frac{(4-n)}{3n}\varepsilon + \frac{19n^3-345n^2+750n-424}{81n^3}\varepsilon^2 \tag{3.15}$$

$$y_\Delta^{RC} = \frac{(n-4)}{6(n-2)}\varepsilon + \frac{(-767n^4+27\,332n^3-141\,024n^2+256\,256n-155\,648)}{5184(7n-16)(n-2)^3}\varepsilon^2 \tag{3.16}$$

of which the last two are new results of the present study. All of these exponents vanish at $n = n_R = 4-4\varepsilon+O(\varepsilon^2)$, and here also the fixed points PH and RH coincide as do PC and RC. For $n > n_R$ the pure fixed points are stable, while for $n < n_R$ they are unstable. Also, for $n > n_R$, the random fixed points are at negative, unphysical values of Δ. Thus we arrive at a similar conclusion to that above, namely that for the same ranges of values of n and d, random impurities are either relevant or irrelevant to both isotropic and cubically anisotropic systems. This agrees with the criterion of Harris (1974), since the specific heat exponent α changes sign at n_R for both isotropic and cubic systems.

The flow diagrams which result from these considerations are shown in figure 3. From this we see that, in the ranges $n < n_R$, $n_R < n < n_C$, $n > n_C$ respectively, the random isotropic, pure isotropic and pure cubic fixed points are absolutely stable. At given values of n and d, a system with coupling parameters exactly on one of the special trajectories in figure 3 will be mapped by renormalisation into the more stable of the two fixed points joined by the trajectory, which will accordingly control its critical behaviour. These are, however, very special conditions, and almost all trajectories eventually flow into the absolutely stable point, which therefore controls the asymptotic behaviour of almost all systems to which our model is applicable. (We are excluding, for the present, those regions of the parameter space which lie outside the domain of attraction of the entire system of fixed points.) This does not, of course, necessarily imply that the true asymptotic behaviour will become apparent in an experimentally accessible range of temperatures, nor that behaviour characteristic of an unstable fixed point may not be approximately realised in some intermediate range of temperatures.

3.3. Singularities and complex eigenvalues

For practical purposes, one would like to know the values of n_C and n_R at $d=3$ or $\varepsilon=1$. The estimates $n_C = 4-2\varepsilon = 2$ and $n_R = 4-4\varepsilon = 0$ appear suspect both because of the large first-order corrections and in view of the singularities at $n=1$ and $n=2$ associated with the random fixed points. There are, moreover, further singularities in the exponents (3.12), (3.14) and (3.18) at $n=\frac{8}{5}$ and $n=\frac{16}{7}$, whose origin we now explain. The random isotropic fixed point RH may be considered in the absence of cubic perturbations. Its two eigenvalues corresponding to directions in the $u-\Delta$ plane are eigenvalues of a 2×2 matrix, and are therefore the roots of a quadratic secular equation, which may be written as

$$y_\pm^{RH} = \tfrac{1}{2}(C \pm D^{1/2}) \tag{3.17}$$

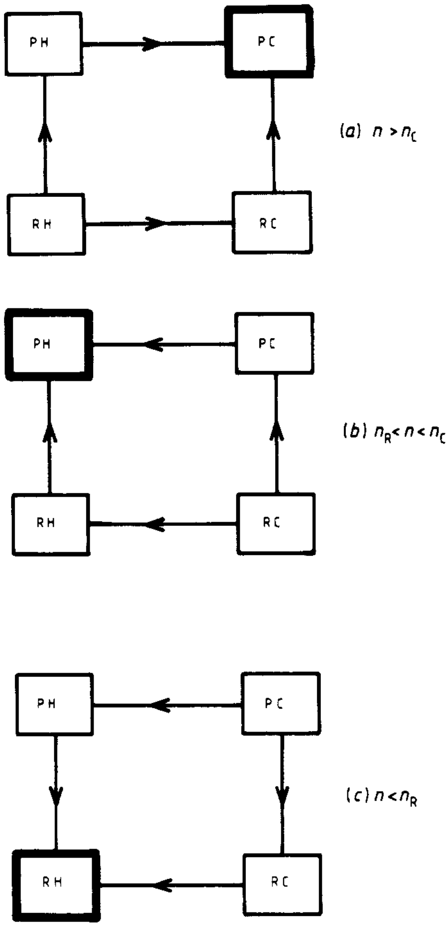

Figure 3. Schematic flow diagrams for (a) $n > n_C$, (b) $n_R < n < n_C$ and (c) $n < n_R$. In each case, a bold box denotes the most stable fixed point.

where

$$C = -\frac{3n}{4(n-1)}\varepsilon + \frac{(285n^3 - 640n^2 + 1088n - 256)}{512(n-1)}\varepsilon^2 + O(\varepsilon^3) \tag{3.18}$$

$$D = \left[\frac{5n-8}{4(n-1)}\right]^2 \varepsilon^2 - \frac{(1495n^4 - 9344n^3 + 18\,240n^2 - 11\,008n + 2048)}{1024(n-1)^4}\varepsilon^3 + O(\varepsilon^4) \tag{3.19}$$

When, for $n \neq \frac{8}{5}$, (3.17) is expanded in powers of ε, the series for y_+^{RH} reproduces the expression y_Δ^{RH} given in (3.14), while y_-^{RH} gives the exponent

$$y_u^{RH} = -\varepsilon + \frac{(365n^3 - 1488n^2 + 2112n - 512)}{128(5n-8)(n-1)^2}\varepsilon^2 + O(\varepsilon^3) \tag{3.20}$$

Apart from a minor numerical discrepancy in (3.20), these agree with the expressions given by Yamazaki (1978). Obviously, however, the expansion is legal only when the discriminant is positive. Proceeding as before, we find that D vanishes on the curve

$$n = n_\pm^{RH} = \tfrac{8}{5}[1 \pm 3(\varepsilon/5)^{1/2} + O(\varepsilon)] \tag{3.21}$$

and that the exponents $y_{\pm}^{\rm RH}$ are complex for $n_{-}^{\rm RH} < n < n_{+}^{\rm RH}$. For this range of n, the fixed point is therefore focal. Let us parametrise the interior of this region by

$$n = \tfrac{8}{5} + \tfrac{24}{25}\lambda\varepsilon^{1/2} + {\rm O}(\varepsilon) \tag{3.22}$$

with $|\lambda| < \sqrt{5} = 2.236\ldots$. Then the real part of the exponents $y_{\pm}^{\rm RH}$ is

$$\mathrm{Re}(Y_{\pm}^{\rm RH}) = \tfrac{1}{2}C = -\varepsilon[1 - \lambda\varepsilon^{1/2} + {\rm O}(\varepsilon)]. \tag{3.23}$$

For sufficiently small ε this is negative, so that the fixed point is stable. When ε is greater than $\frac{1}{5}$, this expression may become positive, if taken at face value. Although there are no grounds for confidence that $\mathrm{Re}(Y_{\pm}^{\rm RH})$ really changes sign in the neighbourhood of $\lambda = \varepsilon^{-1/2}$, let us for the sake of argument restrict attention to smaller values of λ, for which $n < \frac{64}{25} = 2.56$. The situation is depicted in figure 4. On the one hand, extrapolation from the region of $n = 4$, $\varepsilon = 0$ indicates that RH is unstable for $n > n_{\rm R} = 4 - 4\varepsilon + {\rm O}(\varepsilon^2)$, that is above the line (a). On the other hand, we have just argued that for $n_{-}^{\rm RH} < n < n_{+}^{\rm RH}$, that is, inside curve (b), the exponents are complex with, at least below the line (c) where $n = 2.56$, negative real parts, which means that the fixed point should be stable. We see from figure 4 that when ε is greater than about 0.35, there is a region bounded above by (c) and below by (a) in which these conclusions are contradictory. Moreover, (a) was obtained by assuming that the expansion (3.14) was legal, which is not true inside (b). The disappointing conclusion from this is that the ε expansion at the order we have considered yields unambiguous results on the stability of the random isotropic fixed point only for $\varepsilon < 0.2$, say, where (a) lies safely above (b) while $\mathrm{Re}(y_{\pm}^{\rm Rh})$ is unambiguously negative inside (b).

Similar remarks apply to the random cubic fixed point RC. The exponents (3.12) and (3.16) have singularities at $n = \frac{8}{5}$ and $n = \frac{16}{7}$. Since the extrapolation from $n = 4$

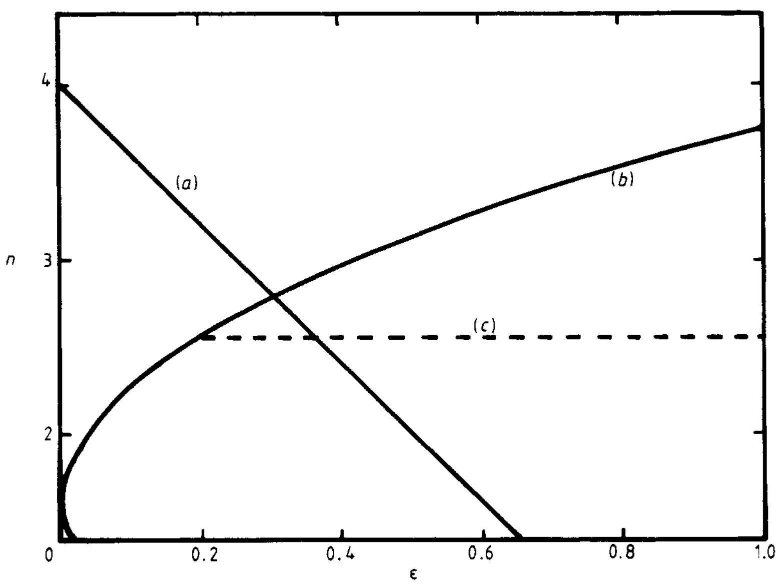

Figure 4. Stability properties of the random Heisenberg fixed point. Near $n = 4$, $\varepsilon = 0$, the fixed point is stable below the line (a), namely for $n < n_{\rm R} = 4 - 4\varepsilon$. Within the parabola ($b$), the eigenvalues are complex. The significance of line (c) is explained in the text.

fails in any case at $n=2$, we consider only the latter. The three eigenvalues of RC are the roots of a cubic secular equation,

$$y^3+r(n,\varepsilon)y^2+s(n,\varepsilon)y+t(n,\varepsilon)=0 \tag{3.24}$$

whose coefficients are

$$r=\varepsilon+\frac{(-3642n^3+21\,732n^2-43\,104n+28\,416)}{5184(n-2)^3}\varepsilon^2+O(\varepsilon^3) \tag{3.25}$$

$$s=-\rho^2\varepsilon^2+\frac{(-1783n^4+8368n^3-1968n^2-29\,696n+29\,696)}{15\,552(n-2)^4}\varepsilon^3+O(\varepsilon^4) \tag{3.26}$$

$$t=-\rho^2\varepsilon^3+\rho\frac{(1457n^3-15\,036n^2+39\,264n-30\,464)}{5184(n-2)^3}\varepsilon^4+O(\varepsilon^5) \tag{3.27}$$

with

$$\rho=\frac{(n-4)}{6(n-2)} \tag{3.28}$$

The lowest-order approximation to (3.24) factorises as $(y^2-\varepsilon^2\rho^2)(y+\varepsilon)=0$, and the first corrections to the roots thus obtained may be straightforwardly found, giving y_v^{RC}, y_Δ^{RC} as in (3.12) and (3.16) respectively, together with

$$y_u^{\mathrm{RC}}=-\varepsilon+\frac{(33\,658n^5-340\,308n^4+137\,811n^3-2783\,488n^2+2795\,520n-1114\,112)\varepsilon^2}{1728(5n-8)(7n-16)(n-2)^3}. \tag{3.29}$$

Obviously, these expansions are illegal when (3.24) has a pair of complex roots, and the poles in the $O(\varepsilon^2)$ terms are, as before, a warning of this. The condition for a pair of roots to be complex is $q^2>p^3$ where $p=r^2-3s$ and $q=\frac{1}{2}(2r^3-9rs+27t)$, and we find that this occurs for $n_-^{\mathrm{RC}}<n<n_+^{\mathrm{RC}}$ where

$$n_\pm^{\mathrm{RC}}=\tfrac{16}{7}[1\pm\tfrac{3}{14}(6\varepsilon)^{1/2}+O(\varepsilon)]. \tag{3.30}$$

On defining λ by

$$n=\tfrac{16}{7}+\tfrac{24}{49}\lambda\varepsilon^{1/2}+O(\varepsilon) \tag{3.31}$$

we obtain for the real part of the exponents corresponding to y_u^{RC} and y_Δ^{RC}

$$\mathrm{Re}\,(y_\pm^{\mathrm{RC}})=-\varepsilon[1-\lambda\varepsilon^{1/2}+O(\varepsilon)]. \tag{3.32}$$

The analogue for the random cubic fixed point of figure 4 is given as figure 5, where the line (c) corresponding to $\lambda\varepsilon^{1/2}=1$ is at $n=\frac{136}{49}=2.776$.

3.4. Special fixed points and $\varepsilon^{1/2}$ expansions

We noted above that in the pure system, the model with $u=0$ corresponds to a set of n decoupled Ising-like systems. It is clear that, although the impurities couple equally to all n components, the quenched average of a correlation function for one component remains decoupled from the remainder. It is therefore true to all orders that the plane $u=0$ is identical to that corresponding to the random Ising system studied by Khmel'nitskii (1975) and Shalaev (1977). It contains a pure Ising fixed point PI at $\Delta=0$ and a random Ising fixed point RI at $\Delta>0$ (together with the unphysical mirror image of the

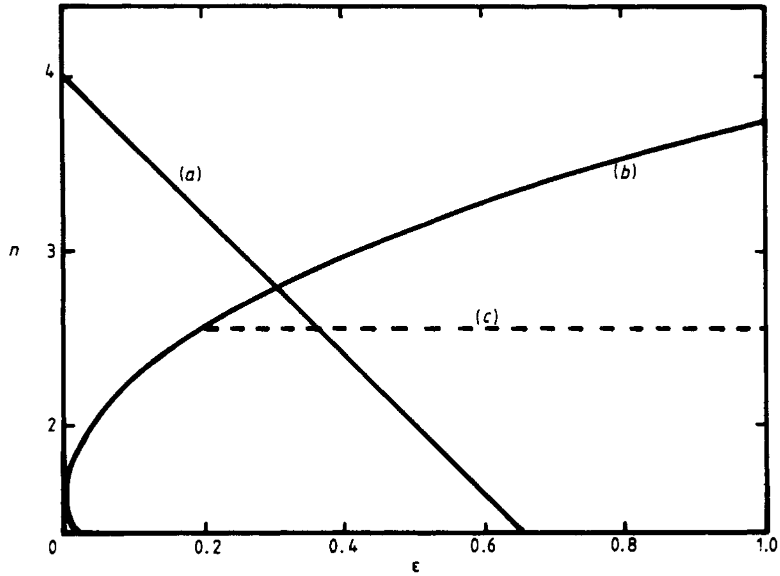

Figure 5. Stability properties of the random cubic fixed point. Curves have the same meaning as in figure 4.

latter at $\Delta < 0$). Owing to a degeneracy of the renormalisation group equations, the random fixed point and its exponents must be obtained as an expansion in $\varepsilon^{1/2}$, and only the leading terms of the expansion can be obtained from a two-loop calculation. Within the plane $u = 0$, the random fixed point is stable. Its exponents are

$$y_v^{\mathrm{RI}} = \mathrm{O}(\varepsilon) \tag{3.33}$$

$$y_\Delta^{\mathrm{RI}} = -2\left[\frac{6\varepsilon}{53}\right]^{1/2} + \mathrm{O}(\varepsilon) \tag{3.34}$$

but according to Shalaev (1977), a three-loop calculation yields a negative value for y_v. Both fixed points are, at least for sufficiently small ε, unstable to the isotropic interaction governed by u. Aharony (1973) gives

$$y_u^{\mathrm{PI}} = \tfrac{1}{3}\varepsilon - \tfrac{19}{81}\varepsilon^2 + \mathrm{O}(\varepsilon^3) \tag{3.35}$$

and we find

$$y_u^{\mathrm{RI}} = 2\left[\frac{6\varepsilon}{53}\right]^{1/2} + \mathrm{O}(\varepsilon). \tag{3.36}$$

When $n = 1$, as observed earlier, there is really only a single interaction of strength $(u + v)$ and the entire parameter space is equivalent to the $v - \Delta$ plane.

Finally, at $n = 2$, a similar degeneracy, first noted by Yamazaki *et al* (1985) affects the random cubic fixed point. We obtain for the coordinates of the fixed point RC($n = 2$)

$$\begin{aligned} u^* &= \frac{3}{2K_4}\left[\frac{\varepsilon}{318}\right]^{1/2} + \mathrm{O}(\varepsilon) \\ \Delta^* &= \frac{6}{K_4}\left[\frac{\varepsilon}{318}\right]^{1/2} + \mathrm{O}(\varepsilon) \\ v^* &= -\frac{1}{K_4}\left[\frac{\varepsilon}{318}\right]^{1/2} + \mathrm{O}(\varepsilon) \end{aligned} \tag{3.37}$$

and there is also an unphysical fixed point at $(-u^*, -\Delta^*, -v^*)$. The eigenvalues of the physical fixed point are

$$y_u = -2\varepsilon + O(\varepsilon^{3/2}) \tag{3.38}$$

$$y_v = 12\left[\frac{\varepsilon}{318}\right]^{1/2} + O(\varepsilon) \tag{3.39}$$

$$y_\Delta = -12\left[\frac{\varepsilon}{318}\right]^{1/2} + O(\varepsilon). \tag{3.40}$$

Apart from a small numerical discrepancy, these agree with the results of Yamazaki *et al* (1985) (their values involve the number 317 rather than 318!). The signs of these exponents are in accord with the schematic flow diagram of figure 3(*c*) which again suggests that this fixed point may in fact be the limit as $n \to 2$ of RC.

4. Conclusions

We have obtained recursion relations valid to second order of the ε expansion for the n-component Landau-Ginzburg-Wilson model containing cubic anisotropy and quenched random impurities. Except in some special cases discussed in § 3.4, the asymptotic critical behaviour is determined by the outcome of a competition between four fixed points, the pure isotropic and cubic fixed points PH and PC and their random counterparts RH and RC. The random cubic fixed point and its critical exponents (3.1)-(3.5) are novel results of the present work. Systematic use of the ε expansion shows that in both isotropic and anisotropic systems, random impurities are relevant for $n < n_R = 4 - 4\varepsilon + O(\varepsilon^2)$, while in both pure and impure systems, cubic anisotropy is relevant for $n > n_C = 4 - 2\varepsilon + O(\varepsilon^2)$. In the three intervals $n < n_R(d)$, $n_R(d) < n < n_C(d)$ and $n > n_C(d)$, the most stable fixed points are respectively RH, PH and PC.

The application of these results to three-dimensional physics is difficult because of the poor convergence of the series. Indeed, we identified in § 3.3 singularities in the ε expansions for the eigenvalues of the random fixed points (these are associated with the onset of focal behaviour) which make the extrapolation especially hazardous. It is of course possible to obtain estimates of $n_R(3)$ and $n_C(3)$ simply by setting $\varepsilon = 1$ in the expressions for appropriate eigenvalues, and determining the values of n at which the resulting functions vanish. The results are mutually inconsistent and not particularly reliable, but for completeness we record the values obtained in this way from those eigenvalues which do not have obvious singularities. From the vanishing of y_Δ^{PH} and y_Δ^{PC}, we obtain respectively the estimates 1.522 and 2.185 for $n_R(3)$. The vanishing of y_v^{PH} and y_v^{PC} yield respectively the estimates 2.0 and 2.282 for $n_C(3)$. The latter estimate for $n_C(3)$ is very close to that obtained by other methods by Yalabik and Houghton (1977), but this is almost certainly fortuitous.

It is of course well known that a large part of the uv plane for pure systems lies outside the domains of attraction of all identifiable fixed points. It is believed that the runaway of renormalisation group trajectories in this region corresponds to a fluctuation-induced first-order transition, and a detailed construction of the free energy which confirms this has been given for the special case $n = 2$ by Rudnick (1978). The essential property of the flow diagram which facilitates this analysis is that every trajectory which runs away eventually enters a region of the uv plane which corresponds to an unstable Hamiltonian. We have checked numerically, for a representative set of parameter values, that this property is preserved when random impurities are introduced, so there would seem to be no impediment in principle to extending the argument of Rudnick to the more general models considered here. On the other hand, we have not succeeded either analytically or numerically in locating the two-dimensional surface of tricritical points which divides first- from second-order transitions. This problem has been investigated by Yamazaki *et al* (1985) for the case $n = 2$, but their results are valid only for very small values of Δ.

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Note added in proof. Since submitting this work for publication, we have learned of recent papers by Yamazaki *et al* (1986a, b) in which a random cubic fixed point is also identified. These papers also consider the effects of extended impurities and purely relaxational dynamics. Their numerical results for static critical exponents do not agree with those given here.

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